Int. J. Solids Structures, 1967, Vol. 3, pp. 259 to 283. Pergamon Press Ltd. Printed in Great Britain

THEORETICAL INVESTIGATIONS ON THE INSTABILITY OF STEEL FRAMES IN THE ELASTO-PLASTIC RANGE†

MICHELE PAGANO and GIOVANNI MORABITO

Institute of the Theory of Structures, University of Naples, Italy

Abstract—This paper deals with the initial results of theoretical and experimental research concerning the behaviour of a steel *I*-beam frame structure under a generic scheme of proportionally variable and horizontal loads.

NOTATION

D_i	$h_i^2/6EI_0$
E	elastic modulus of the material
F_i	sum of horizontal forces acting above floor i relative to the service load
h_i	height of storey i
i	order of numeration of storeys ($i = 1 \ldots r$) and thus also of the unknowns corresponding to displacements, in the system of equations
K	generic node
I_0	moment of inertia of the section of the bar chosen as the unit of length
$m^{(k)}$	number of beams meeting at joint K
$M = \sum_K (m^{(k)}-1)$	
n	number of phases ($n = 1, 2 \ldots$)
$n-1$	total number of plastic hinges in the nth configuration; by the application of $\Delta\alpha^{(n)}q$ the nth hinge is formed
N_i	sum of normal forces in all stanchions of storey i for the service load configuration
q	symbol for the service load configuration
$q^{(n)} = \sum_1^n q$	symbol for the highest load applied in the nth phase
r	total number of floors
t	total number of nodes
$X_i^{(n)}(X_i^{(s)})$	value of increase of displacement of storey i in the nth phase (sth), (disregarding factor D_i)
X_j	unknown bending moments (j = number of the order of unknowns in the system of equations)
$X_j^{(n)}$	increase in unknown bending moment in section j in nth phase ($j > r$)
$\alpha^{(n)}$	factor defining the total load applied in the nth phase
$\Delta\alpha^{(n)}(\Delta\alpha^{(s)})$	multiplier of the service load configuration which defines the maximum stress which the structure can bear in the nth (sth) phase
$\Delta\sigma_p$	variation of stress in the pth plastic hinge ($p = 1, 2 \ldots n-1$)
$\varphi_{ak}(\varphi_{bk})$	rotation of the end of a generic bar $a(b)$ anchored at node K
$\varphi_p^{*(n)}$	rotation of pth hinge in the nth phase

INTRODUCTION

LEAVING to a later paper the results of experimental investigation still being carried out, this study presents through the application of the four-moment equation a technique for

† The present study has been carried out under the auspices of the Centro Studi per l'edilizia of the University of Naples, supported by the Italsider Company and the Consiglio Nazionale delle Ricerche.

fully evaluating for a structure the variation of stresses and displacements introduced by the destabilizing effect of vertical loads assuming an ideal elasto-plastic behaviour of the material and the idealization of the cross-section of the bars as two concentrated masses. The computerized calculation procedure illustrated here yields a quantitative evaluation of the phenomenon in its entirety from a condition of full elasticity to final collapse, within a time sufficiently short for practical applications.

The structure, initially highly hyperstatic, is progressively weakened, as the load increases, by the formation of successive plastic hinges, and shows for a value of the load, which is defined as critical, a phenomenon of "snapping". Subsequently, under conditions of instability and with decreasing loads, the structure continues to distort and additional plastic hinges are formed, which reduce it to a kinematic mechanism.

The preliminary hypotheses and the approximations adopted are also discussed. In conclusion, it is thought that for structural steel frames, the procedure proposed will provide an acceptable means of calculation, both from the standpoint of theoretical formulation and practical application. It is hoped that experimental results to come will also prove it to be conservative. A selected bibliography on this subject has been included.

1

The calculation of plastic collapse presents difficulties when undertaken in general terms from a purely theoretical viewpoint as regards the structural configuration, the shape of the bars (perhaps curvilinear and of variable cross-section), their cross-section and the material of which they are made.

In practice, however, considerable simplifications can generally be made when properties mentioned above are specified. In other words, especially important structures can always be specially designed because in such cases the work-time and design effort are justified, even though they are greater than normal.

In an earlier study [1] the following restrictive hypotheses have been formulated:

(a) The frame consists of rectangular units built up from rectilinear bars of constant section and vertical stanchions.

(b) The structure is considered continuous and, for the service loading, the ratio of the normal force to the Euler load for every bar (P/P_E) is small (less than 0·1). This hypothesis should, in general, be satisfied in framed structures in order to prevent excessive deformability.

(c) The material used is still with elasto-plastic characteristics presented in the usual bilinear form, and all loads are increased (or decreased) proportionally, by a common load factor α.

(d) The yield limit, experimentally obtained, is not defined by means of a single value but by a range of values which contains all the test findings on specimens. Such a hypothesis is appropriate when using an iterative procedure [1] with load increments determined by trial, since less precision is required in determining the magnitude of such increments.

(e) The limit moment corresponding to the formation of a hinge condition is independent of the normal force variations subsequent to its formation.

(f) The deformations due to shear and to normal forces, the latter being considered both as axial shortening and as a reduction in flexural rigidity, are negligible.

(g) Each hinge forms when the strain in the stanchions due to normal force and bending moment or the strain in the horizontal bars due to bending moment alone attains a value within the yield belt, neglecting the effect of the shear force (Fig. 5)†. Also ignored are local concentrations of stress corresponding to points of load application and connections at nodes, as well as any other possible local effects such as internal rolling stresses and faults, material flaws, eccentricities of construction, etc.
(h) The cross-section is idealized by two concentrated masses. Included in this hypothesis, it is felt, are most of the sections used in steel structures, e.g. I-beams and box beams.
(i) The hinges are localized in one cross-section, taking account of its shape. The eccentricity of the hinge, depending on the simultaneous presence of a bending moment and a normal force, has been ignored.

Under these conditions the limit of elasticity is reached throughout a given section at the same time. The present study, from a theoretical point of view, is not substantially affected if the ultimate stress in the bar is reached at the same time throughout any other part of its length.‡

Through such hypotheses, there was proposed in a previous study [1] an iterative procedure to analyse, step-by-step, the stresses and deformations of a structure in the elasto-plastic range. The destabilizing effects of vertical loads were also taken into account (Fig. 1) and there was thus obtained a critical value α^* of the load factor for a hyperstatic structural design. The results are substantially different from those of the "rigid-plastic" theory which, apart from any destabilizing effect, identifies the collapse with the value of the critical load factor which makes the structure partially or completely unstable.

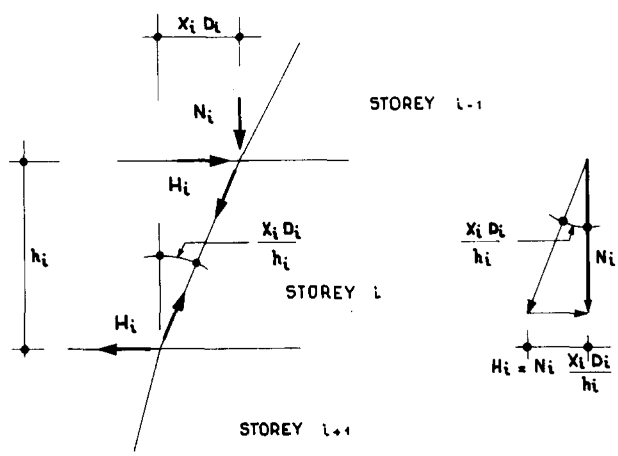

FIG. 1 The forces H_i represent the destabilizing effect of the total vertical load N_i acting on floor i, due to the inclination $X_i D_i/h_i$ of storey i.

† In order to take into account the phenomena of hyperelasticity, it is convenient to assume values for the limit of elasticity based on physical flexural tests of each bar.

‡ This case would correspond to a constant moment throughout a simple bar, the normal force being held constant.

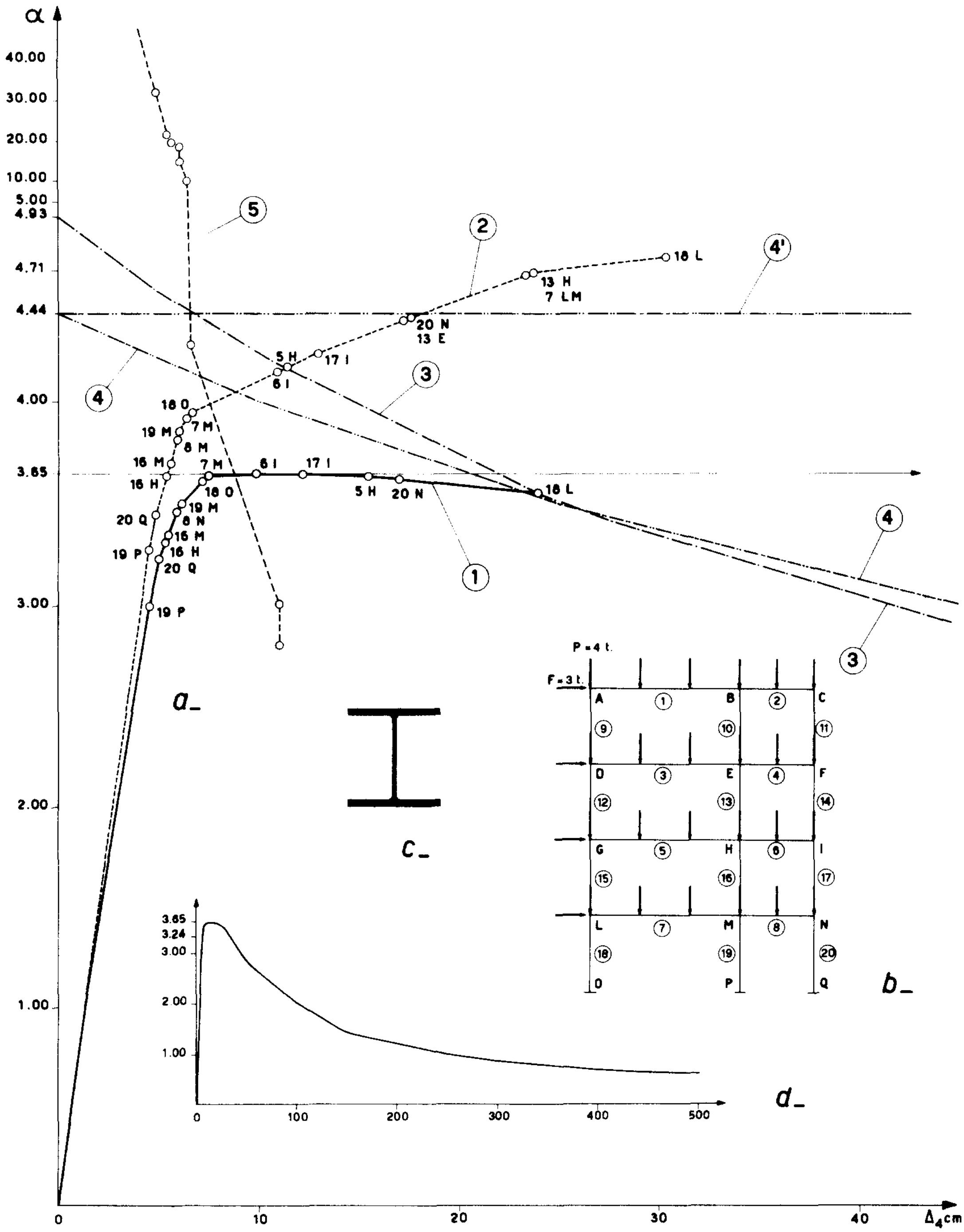

FIG. 2(a). Curve (1) represents the link between the load factor and the displacement of the bottom floor (Δ_4) according to the new theory; Curve (2) represents this same link according to the rigid-plastic theory; Curve (3) is relative to the state of instability of the bottom storey ($i = 4$), in continuation of curve (1); Curve (4) is analogous to curve (3), but takes into account the effect of normal stress; Curve (4') is again analogous to curve (3), but does not take into account the destabilizing effect of vertical loads and is continued up to the axis of the ordinates; Curve (5) represents the critical load factor (in the elastic range) referred to curve (2). FIG. 2(b). Configuration of frame with service loads ($\alpha = 1$). FIG. 2(c). Constant cross-section of beams HE_b 300. FIG. 2(d) Curve (1) of Fig. 1(a) shown to different scale.

The two theories are compared in Fig. 2. The abscissa represents the lateral displacement of the bottom storey (Δ_4) and the ordinate represents the load factor (α).

Curve (1) refers to the study mentioned above. It shows that the frame, having attained the value $\alpha^* = 3{\cdot}65$ after several load increments, subsequently deforms according to the descending arc of the curve (unstable) corresponding to load decrements.

Curve (2), on the contrary, relative to rigid-plastic theory, shows only an ascending line in which the structures bear successive load increments and reaches a final value which is much higher ($\alpha^* = 4{\cdot}71$) than the one shown in the previous curve (1).

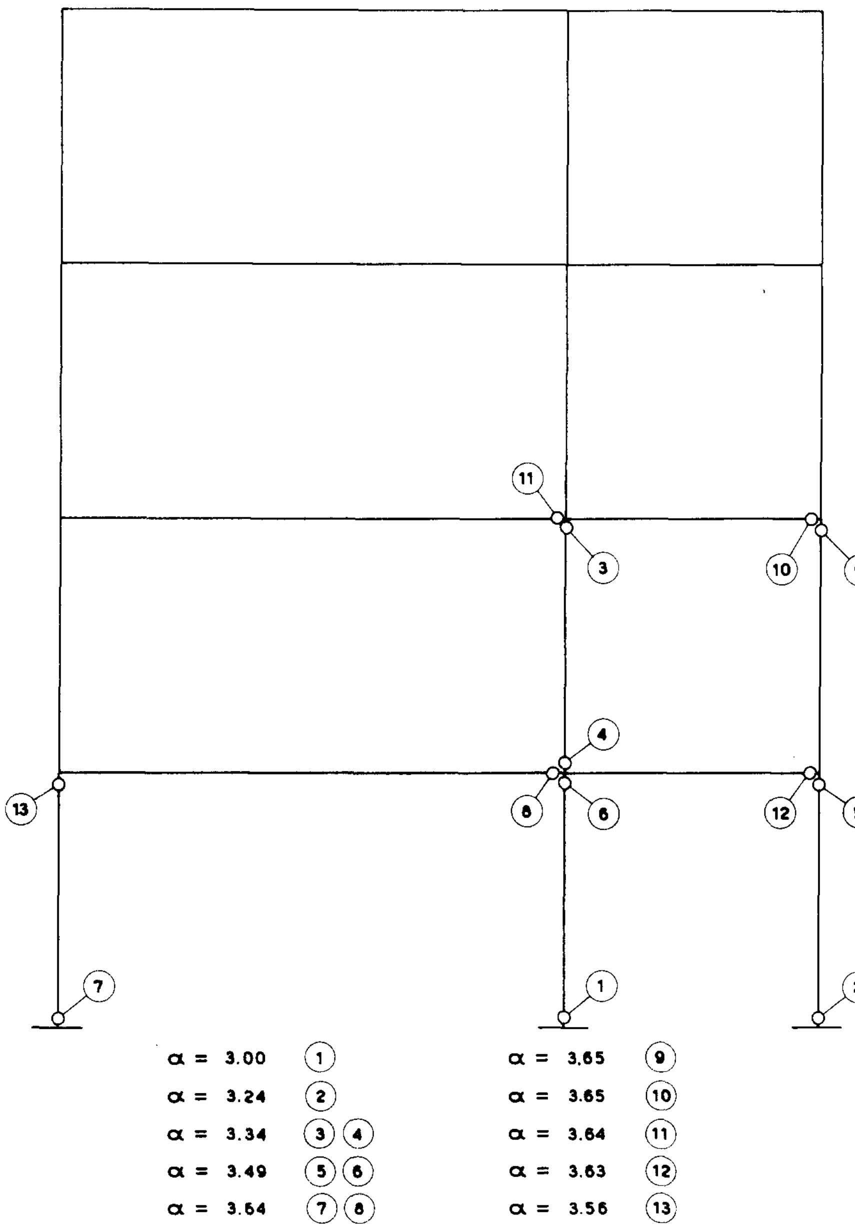

FIG. 3(a)

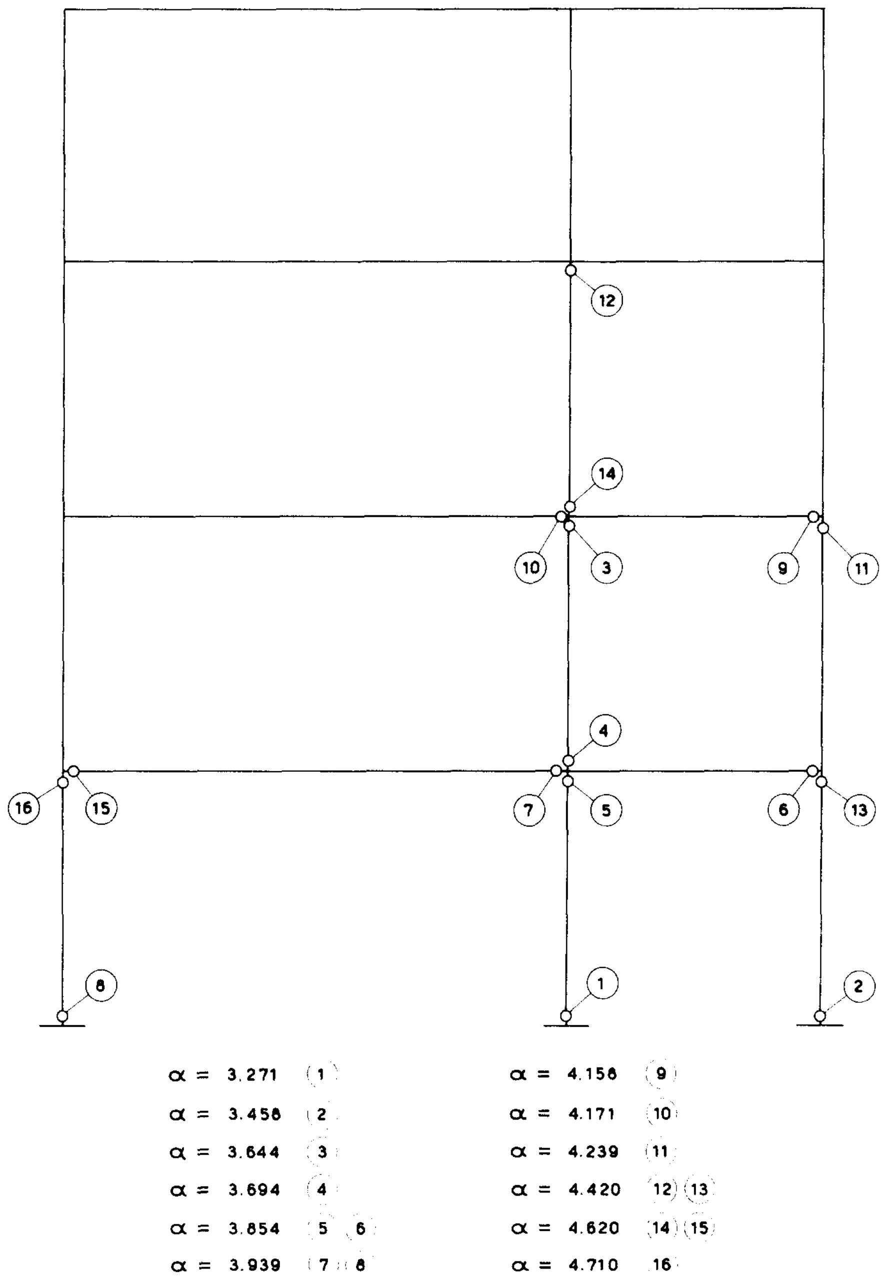

FIG. 3(b)

FIG. 3. Plastic hinges formed in the frame in the order corresponding to the numeration. The value of the load factor is shown, (a) = new procedure, (b) = rigid-plastic theory.

Along curves (1) and (2) are shown the hinges which are gradually formed in the frame [Figs. 3(a) and (b)]. They differ, in the two theories, both in number and distribution.

Curve (3) illustrates the conditions of equilibrium of the kinematic mechanism which forms at the end of curve (1) and its equation is:

$$\alpha = \frac{\Sigma M_{ik}^*}{F_4\sqrt{(h_4^2 - D_4^2 X_4^2)} + N_4 D_4 X_4}$$

in which ΣM_{ik}^* represents the sum of the limit moments of the sections of the extremities of the columns of the bottom floor and $D_4 X_4$ represents the horizontal component of the displacement of the first floor. If in the evaluation of the limit moment, account is taken of the existing normal stress, then the values of M_{ik}^* are all different from one another. In describing the curve (3), the values of M_{ik}^* have been assumed constant and correspond to the value of the normal stress that is present in every section at the end of curve (1).

If on the other hand, account is taken of the variability of the normal stress, then curve (4) is obtained, whose equation is:

$$\alpha = \frac{3HN_p}{N_4[D_4X_4 + (H/h_4)\sqrt{(h_4^2 - D_4^2X_4^2)}] + F_4[\sqrt{(h_4^2 - D_4^2X_4^2)} - (H/h_4)D_4X_4]}$$

where N_p is the "fully plastic thrust" and H is the height of the cross-section. [Fig. 2(c).]

Furthermore, curve (4), which is in fact a horizontal, is derived from the preceding relationship, provided the destabilizing effect of the vertical loads is ignored. In this case the first term of the denominator vanishes.

Curve (5) traces the values of the global critical load factors in the elastic range for the structural schemes which are gradually formed, in accordance with the rigid-plastic theory shown for curve (2). For each structural scheme of curve (2), real hinges are substituted for plastic ones and the material itself is considered as ideally elastic.

The values of these critical factors can be found by reducing to zero the determinant of the system of equations corresponding to the successive schemes of the structure.

In [2] the calculation procedure which was carried out with the aid of an electronic computer is shown in detail. For the initial structural scheme, which is very rigid, the value of the critical factor in the elastic range proves to be very high (over 40). For the successive schemes, the value decreases with the increase in the number of hinges and, for a value of approximately $\alpha = 4$, curve (5) intersects curve (2). After this intersection curve (2) loses any physical meaning.

It is evident from the comparison that the two theories identify collapse with values of α and of Δ which differ considerably and it is therefore necessary to introduce the phenomenon of instability in the study of the behaviour of the structure. But for practical technical application it appeared necessary to re-examine the hypothesis adopted and also to work out a sufficiently rapid scheme of calculation. Some of the quoted hypotheses can be verified without any further discussion—as for instance (a) and (b)—whereas others involve an approximation whose accuracy must be verified from both the theoretical [(c), (d) and (e)] and the experimental points of view [(f(, (g), (h) and (i)]. Also, the iterative procedure is undoubtedly very laborious and is therefore of little use in practical applications.

Such are the objections that gave rise to a programme of research. The present paper reports the theoretical results achieved to date in this effort, which are as follows:

(a) Formulation and organization of a calculation procedure which could be carried out by an electronic computer and be rapidly put into technical practice. This calculation is based on the four-moment equation and introduces into conditions of horizontal equilibrium the destabilizing effect of vertical loads. The equation system, changing from one deformed shape of the structure to the next, is considered, together with the plasticization conditions, written for all the most stressed sections, and any other conditions which, during the calculation, control the sign of the variations in the rotation of plastic hinges after their formation.

(b) The elimination of the hypothesis admitting the existence of the "yield belt" in materials. Such a hypothesis, though corresponding to a physical reality and being highly useful for the application of the iterative procedure proposed, suffers from the theoretical defect of not being strictly conclusive to the achievement of static conditions of safety, since, once the existence of a yield belt is admitted, it is always necessary for absolute safety, to relate the formation of a plastic hinge to its lowest limit.

(c) Elimination of the simplifying hypothesis which requires the limit moment, corresponding to the formation of a plastic hinge, independently of the normal force variation which occurs after its formation. In general, when the external loads acting on the structure increase, the normal force varies in each plastic hinge and the limit moment is modified according to the link defined by the boundary AB of the elasto-plastic range as shown in Fig. 5(a). In a separate paper a report will be made on experimental research on a model aimed at comparing theory with practical reality and checking the validity of the above-mentioned hypotheses.

2

In relation to the choice calculation process, it is noted that the calculating system, with the possibilities opened up by computerization, must enable the designer to intervene easily when, with increasing load, the structural geometry becomes modified. Further, the choice of one of the two well known groups of methods available for the theoretical analysis of a highly hyperstatic structure (the "force" and "deformation" methods), permits the adoption of a system of equations and the discarding of the procedure of successive approximations.

The method using the "four-moment equation" has been chosen, since this possesses the essential advantage that the unknowns are the effective bending moments at the extremities of the bars, just at the points where the plastic hinges mainly form; furthermore, it is not difficult to write the expression for the bending moments acting along a bar, in terms of the effective bending moment acting at its extremities. This enables one to operate directly on the unknowns, by imposing suitable conditions† for the various structural geometries which vary as stresses are increased, and successive plastic hinges are formed. In fact, when the system of equations corresponding to the initial structural geometry has been written, it is easy to modify it by excluding equations which subsequently lose their validity (because they express congruence conditions which have lapsed) and introducing appropriate equations to express constraints of plasticity for the cross-sections of the frame in which the material has reached the limit of elasticity in the domain (M, N) represented by the straight line AB in Fig. 5(a).

† The advantage of acting directly on the unknowns compensates for the fact that the sets of equations to be solved is greater than that involved in a "deformation" method.

3

For the practical application of the procedure discussed there must be written, for the initial geometry, the equilibrium relations of the translation of the storeys and the rotation of the nodes, as well as all the congruence relationships possible for each node and the expressions of the maximum stresses in those sections where it is anticipated that the elasticity limit will be exceeded. The system of equations, as related to the nth stage, in which the structural system has $n-1$ plastic hinges, is then made up of the following:

(a) *the equilibrium equations of the translation of the storeys*, equivalent to the number of storeys (r)
(b) *the equilibrium equations of the rotations of the nodes*, equivalent to the number of nodes (t)
(c) *the congruence equations*, so selected in each node as to be linearly independent.† This selection is initially free [with $m(m-1)/2$ possibilities in each node], whereas it will become increasingly restricted, in successive stages, for those nodes where plastic hinges form
(d) *a number of plasticization equations*, equivalent to $n-1$, expressing constraint upon any increase of stress in the already plasticized sections
(e) *the conditions which maintain a constant sign of rotation in every plastic hinge, in all the phases after their formation.*

If the value of $\Delta\alpha^{(n)}$ is the maximum load increase bearable by the nth structural configuration, i.e. such as to bring another section to the threshold of plasticity in such a way that the number of hinges goes from $n-1$ to n, then the corresponding analysis may be expressed symbolically in the following way:

$$\Delta\alpha^{(n)}N_iD_i\left(\sum_{s=1}^{n-1} X_i^{(s)}+X_i^{(n)}\right)+\alpha^{(n-1)}N_iD_iX_i^{(n)}$$
$$+\sum_i X_j^{(n)}+\Delta\alpha^{(n)}F_ih_i = 0 \quad (i = 1, 2, \ldots r)(j > r) \tag{1a}$$

$$\sum_K X_j = 0 \qquad \text{(in number equal to } t) \tag{1b}$$

$$\varphi_{aK} = \varphi_{bK} \qquad (a, b\text{: bars forming node } K) \tag{1c}$$

$$\Delta\sigma_p = 0 \qquad (p = 1, 2, \ldots n-1) \tag{1d}$$

$$\varphi_p^{*(n)}-\varphi_p^{*(n-1)} \gtrless 0 \qquad \text{if } \varphi_p^{*(n-1)} \gtrless 0 \tag{1e}$$

The first term of the sum in parentheses in equations of the type (1a), proportional to the rotation of floor i at the end of the preceding stage $(n-1)$, is known.

In the initial stage ($n = 1$), the type (1d) equations are absent, as there are no plasticized sections in the structure.

Only the relationship type (1a) associated with the displacement of floor i, deserves special note, since the destabilizing effect of vertical loads has been introduced. This is the sum of four terms, of which the third and fourth formally represent the usual ones. For the other two it should be observed that in the nth phase the structure is subjected to the new load $q^n-q^{n-1} = \Delta\alpha^{(n)}q$. The first term in the equation takes into consideration the

† The number of equations equals $M-(n-1)$ if the $n-1$ hinges have all formed at the extremities of the bars (see the Notation).

horizontal component of the part of these loads above the floor i, distributed along the slope of the uprights of storey i and the horizontal. The second term takes into account the increase, again assessed horizontally, which the analogous components of vertical load $q^{(n-1)}$, applied in the preceding $n-1$ phases, undergo because of the further increase in the slope of storey i due to the superposition of the additional loads $q^n - q^{n-1} = \Delta\alpha^{(n)}q$, corresponding to the nth phase.

4

Initially, that is in the absence of plastic hinges, system (1) contains, as has been said, only equations of types (1a), (1b), and (1c).

If the loads are increased, the stresses also increase and, in the most highly stressed sections, the material may reach the limit of elasticity. Generally speaking, it is usually easy to detect those sections in which, by reason of structural design and load, plasticization is likely to occur. Such sections are calculated in terms of the maximum stress.

The hypothesis that the section is made up of only two concentrated masses makes the formation of a plastic hinge coincide with the reaching of yield stress. The value of the load factor $\alpha^{(1)}$ which corresponds to this plasticization condition in the first section, provides the maximum load $q^{(1)}$ which the structure can bear under conditions of absolute elasticity. This value is normally calculated on the basis of the current procedure to verify structural stability. The electronic computer is able to determine the stress values in all the sections chosen in terms of α, and hence to provide the value of $\alpha^{(1)}$. The plasticization condition which must be verified is expressed in the following form:

$$M^{(1)}/W + N^{(1)}/A = \sigma_s$$

where $M^{(1)}$ and $N^{(1)}$ are the bending and axial characteristics of the first section to become plasticized, which can be expressed in terms of the unknowns $X_j^{(1)}$, W is the flexural modulus of resistance, and A is the area of the section.

In the following phase, in this plasticized section, no further increases of stress can occur. Therefore by making the stress due to further load increases equal to zero, we obtain the first plasticization equation (1d) to be adhered to which replaces, in the system of equations, the congruence equation now no longer valid. In node K, in fact, we may write only $(m^{(k)}-2)$ congruence equations ($m^{(k)}$ being the number of bars making up node K).†

The plasticization equation will therefore be:

$$\Delta\sigma = M^{(2)}/W + N^{(2)}/A = 0$$

where $M^{(2)}$ and $N^{(2)}$ are the values of the increases in moment and normal force in the plasticized section, again expressed in terms of the unknowns $X_j^{(2)}$.

The procedure to be followed in successive phases is perfectly analogous: the plasticization equations are progressively substituted for congruence equations. Such modifications prove to be very easy by the chosen method of calculation. Load increases sufficient to create further hinges, become progressively smaller. In fact, there occurs for a phase which can be termed "critical", corresponding to a structural scheme which is always hyperstatic, a condition where deformation and stress cease to be the result of increases in load and in

† It is possible that more than one of the equations in the initial system relative to node K in which the hinge has formed, may become inoperative. In this case, $(m-2)$ linearly independent equations must be chosen from among the conditions of congruence relative to the bars where there are no plasticized sections.

which the load values begin to decrease. In other terms, after this condition is reached, the system is satisfied if a load reduction occurs ($\Delta\alpha$ negative), and provides for it a solution that is still balanced and congruous, though unstable. Even a very slight load increase would in fact now involve configurations which are no longer in equilibrium, but tending, by a dynamic succession of movements, towards infinite values of displacements. The procedure then continues until the configuration becomes partially a kinematic mechanism, and hence, as has been previously explained, a single equation of translation equilibrium by itself provides all further configurations of equilibrium of the structure which are unstable. That part of the structure which has become unstable will obviously have deformations affecting the residual part of the structure which is still hyperstatic or, at the most isostatic, and which may therefore be considered as rigid in relation to total deformations.

5

For the procedures set out above, two distinct computer programmes have been formulated which permit the requisite calculations to be made in the SNAP assemblative symbolic language.

With reference to the nth phase in which $n-1$ plastic hinges have already been formed, the value of the load increase $q^n - q^{(n-1)} = \Delta\alpha^{(n)}q$ must be determined and this brings a further nth section to the limit of elasticity.

(a) In the first programme, a value $\Delta\alpha \,.\, q$ of load variation is given to the generic phase n, which is broken-down into suitably small fractions ($\delta\Delta\alpha \,.\, q$).

The total interval $\Delta\alpha \,.\, q$, to be covered by the step $\delta\Delta\alpha \,.\, q$, must be chosen so as to ensure that a further plastic hinge is formed within that interval.

With reference to system (1), the computer first reads the value $\sum_{1}^{n-1} X_i$ for all r storeys of the frame as well as the value $\alpha^{(n-1)}$, which emerges from the preceding phases and which are necessary to construct the known terms b_i (with $i = 1, 2, \dots r$) and the coefficients of the leading diagonal ($C_{11}, \dots C_{rr}$) of the first r equations, respectively, within the matrix of the system. The computer is then fed the extreme values $\Delta\alpha_i$ and $\Delta\alpha_f$ of the interval assigned for $\Delta\alpha$† and with the step $\delta\Delta\alpha$ regarded as most suitable to determine fairly exactly the point of formation of the plastic hinge. After this, the entire system of equations is read off and stored in two different memory cans A and B. On the second one, the method of matrix inversion is used for resolving the system corresponding to the generic value of $\Delta\alpha$ internal to the interval examined. When the results of the interval assigned for $\Delta\alpha$ have been printed out, it is possible, without the computer, to determine the value $\Delta\alpha^{(n)}$ for which the plasticization conditions indicate the attainment of the threshhold of plasticity for a further nth section.

Introducing the corresponding equation type (1d) into the system, in place of type (1c), which is no longer valid, the computer may be made to start the successive phase $(n+1)$.

(b) The second programme allows for the carrying out, by the computer, of all the operations mentioned above, which are necessary not only for solving the single

† Note that the load to be applied to the structure may have a negative value, and therefore the interval to be assigned to $\Delta\alpha$ must also take this possibility into account.

elementary system, but also for checking these equations expressing plasticization conditions. These conditions are then read off and appropriately stored.

In this second programme the computer is given the instructions $\Delta\alpha_i$ and the step $\delta\Delta\alpha$, omitting the interval $\Delta\alpha_i \div \Delta\alpha_f$.

In fact, as soon as a plasticization condition (unequation) is manifested ($\Delta\alpha = \Delta\alpha^{(n)}$), the computer is programmed to stop and print, with obvious advantages over the preceding programme. Machine time is in fact considerably reduced and the resolution of the systems for the interval $\Delta\alpha^{(n)} < \Delta\alpha < \Delta\alpha_f$ and the printing of those relative to the interval $\Delta\alpha_i < \Delta\alpha < \Delta\alpha^{(n)}$ are avoided.

Preparatory and resolving operations for the system are identical with those illustrated for the foregoing programme. Invalid equation detection is additionally inserted, followed by a test to denote the achievement of a condition of plasticity. If the outcome of the test shows that plasticity has in fact come about, then the iterative process stops, as already mentioned, and the computer prints the serial number of the invalid equation which has occurred and prepares to read a further plastic equation, type (1d), to be substituted for the elastic-type equation (1c). A signal from the console resets the system for reading and the resumption of the cycle.

In each phase, therefore, when an invalid equation (unequation) appears, all the data and the value of $\Delta\alpha^{(n)}$ and of $\alpha^{(n-1)}$, and of the bending moments $\sum_1^n X_j$ (which have accumulated in the n phases) are printed.

6

The procedure set out in the foregoing sections was applied to the frame shown in Fig. 4(a), subjected to service loads ($\alpha = 1$). Both above-mentioned programmes were applied and their flow-charts are shown in Figs. 6 and 7.†

The limit stress value was assumed to be 3·100 kg/cm², on the basis of experience gained on a single structural element. Critical conditions occurred after eight phases (corresponding to the formation of eight hinges) for increasing values of α (Fig. 8). Under such conditions the structure is hyperstatic. Subsequently, for decreasing values of α, four other hinges form and a partial kinematic mechanism is reached. In fact, a mechanism forms in the bottom storey, where stresses are higher on account of the particular proportional criterion chosen to indicate more clearly the physical behaviour of the structure and to distinguish from one another, the various factors giving rise to the formation of successive hinges.

The position of the hinges, which are progressively formed, are illustrated in Fig. 8. To each subsequent structural configuration there corresponds an equation system (1) as explained in Section 3.

In Table 1 are shown the most significant values for all twelve phases the structure goes through. Figure 9(a) shows the curves which link the load multiplier to the horizontal displacement of the three floors.

† The calculations were made by the CDC-G 20 type electronic computer of the Electronic Computing Centre of the Engineering Department of the University of Naples.

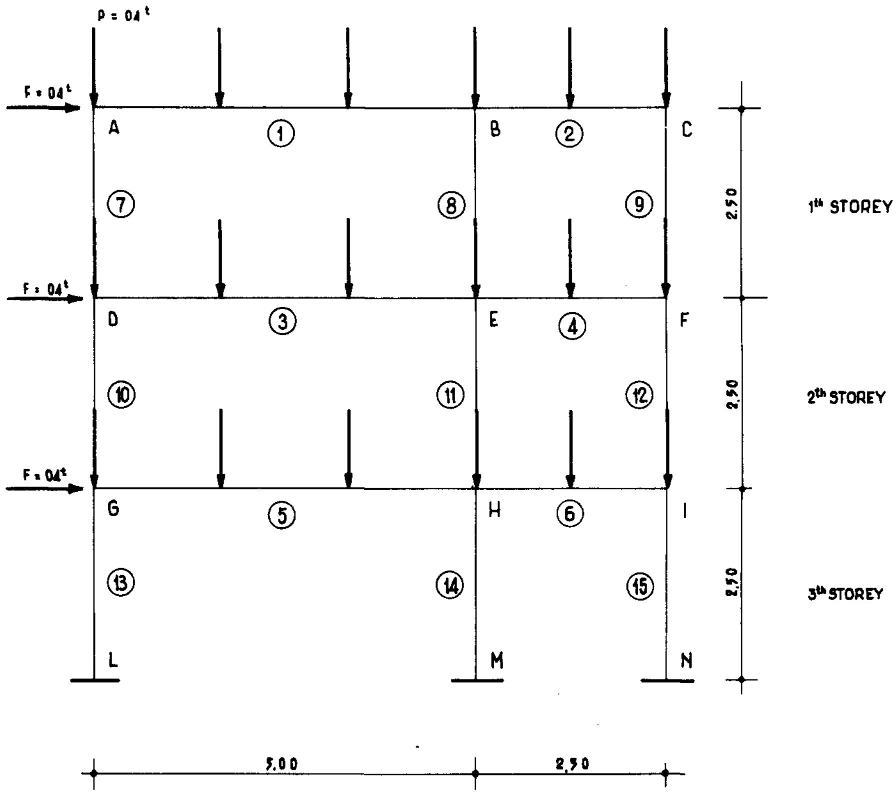

FIG. 4(a). Configuration of the experimental frame and the loads corresponding to $\alpha = 1$. All bars are drawn sections HE_b 100.

$(A = 21{\cdot}26\ \text{cm}^2; \quad I = 431{\cdot}66\ \text{cm}^4; \quad W = 95{\cdot}96\ \text{cm}^3)$

In Fig. 9(b) is shown the rotation of the hinges in relation to sections 25–27 as a function of α. For all the other hinges analogous curves have been found, but the corresponding numerical values are not shown.

Figure 10 shows the "rigid deformation" of the frame in each phase.

7

By way of example, some equations relating to the eighth phase at the beginning of which there are only seven hinges are written. The eighth hinge forms when the load factor has passed from the value 3·8318 (corresponding to the seventh phase) to the value 3·8424.†

The type (1a) equations, relating to the conditions of the translation equilibrium of the storeys, in the system (1) (Section 3) are expressed as follows.

† The equations show the symbol $\Delta\alpha^{(8)}$ which represents the load increase leading to the formation, in the frame, of the eighth plastic hinge. The computer, on the other hand, as already stated and clarified later, analyses numerous values of $\Delta\alpha$ which are variable by a given step.

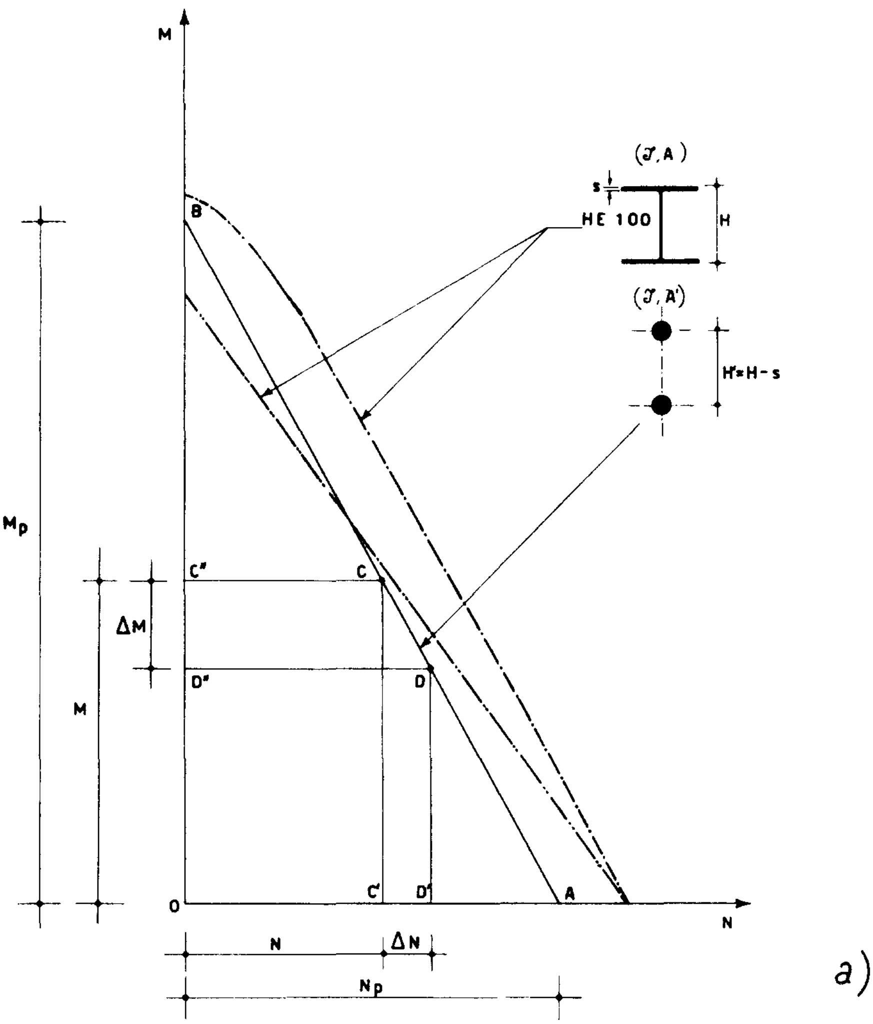

FIG. 5(a). (Plastic range) relative to profile HE_b 100 and to two masses of equivalent moment of inertia concentrated at the ends of the cross-section.

For the top storey we have:

$$\Delta\alpha^{(8)}N_1D\left(\sum_1^7 X_1+X_1\right)+\alpha^{(7)}N_1DX_1+X_8+X_9+X_{10}+X_{11}+X_{12}$$
$$+X_{13}+\Delta\alpha^{(8)}\,.\,F_1\,.\,h_1=0$$

By substituting the values:

$$\sum_1^7 X_1 = 2610{\cdot}4428 \text{ kg-m}$$

$$N_1D = 0{\cdot}027759$$

$$\alpha^{(7)} = 3{\cdot}8318;\quad F_1 = 400 \text{ kg};\quad h_1 = 2{\cdot}50 \text{ m}$$

FIG. 4(b). Photograph showing the experimental frame to be tested, together with the method of applying the load and of measuring the distortion.

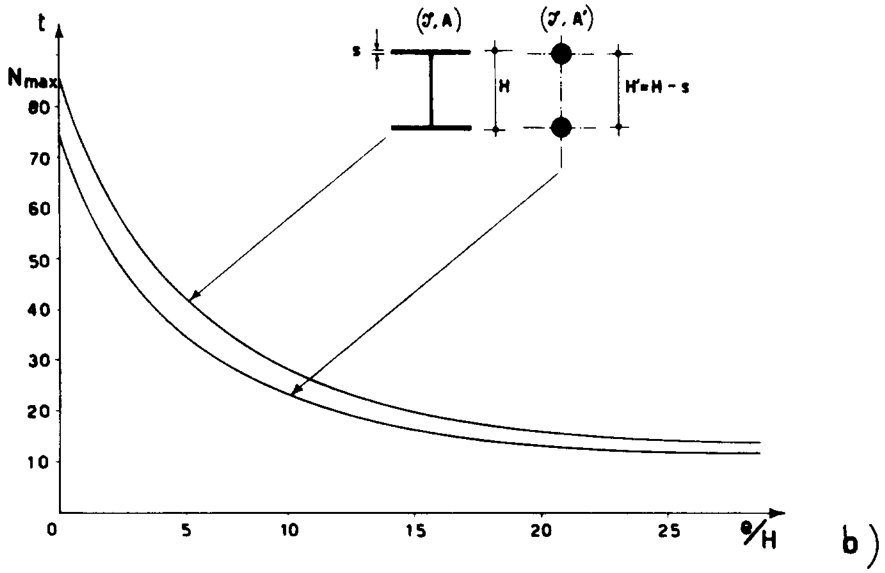

FIG. 5(b). Comparison between the ultimate strength of an HE 100 profile and two masses with equivalent moments of inertia, concentrated at the ends of the cross-section, subjected to an eccentric loading.

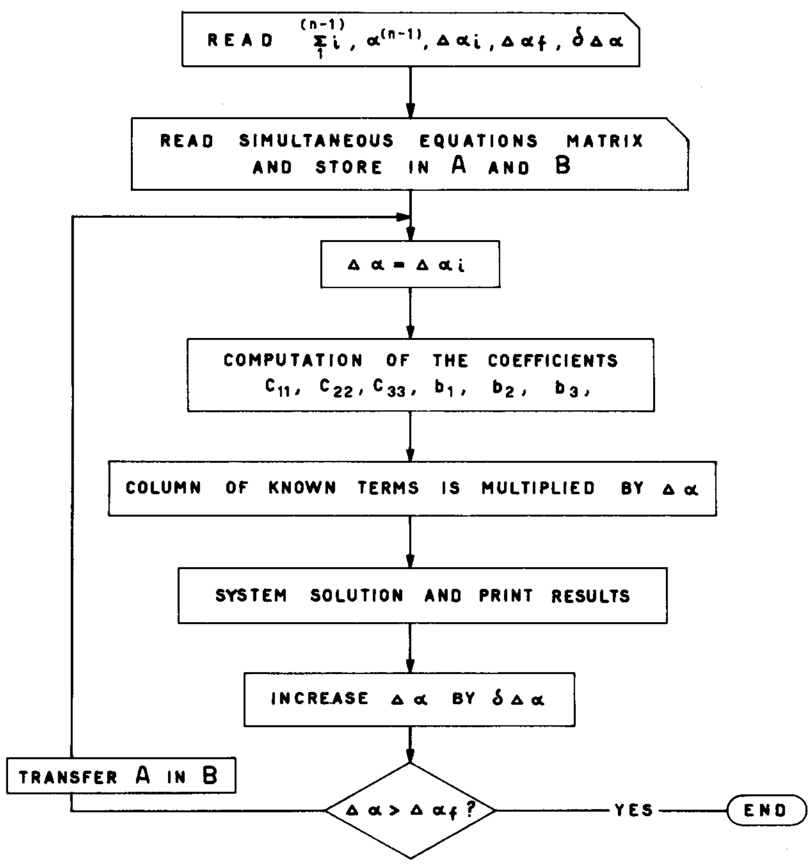

FIG. 6. Flow chart described in Section 7(a).

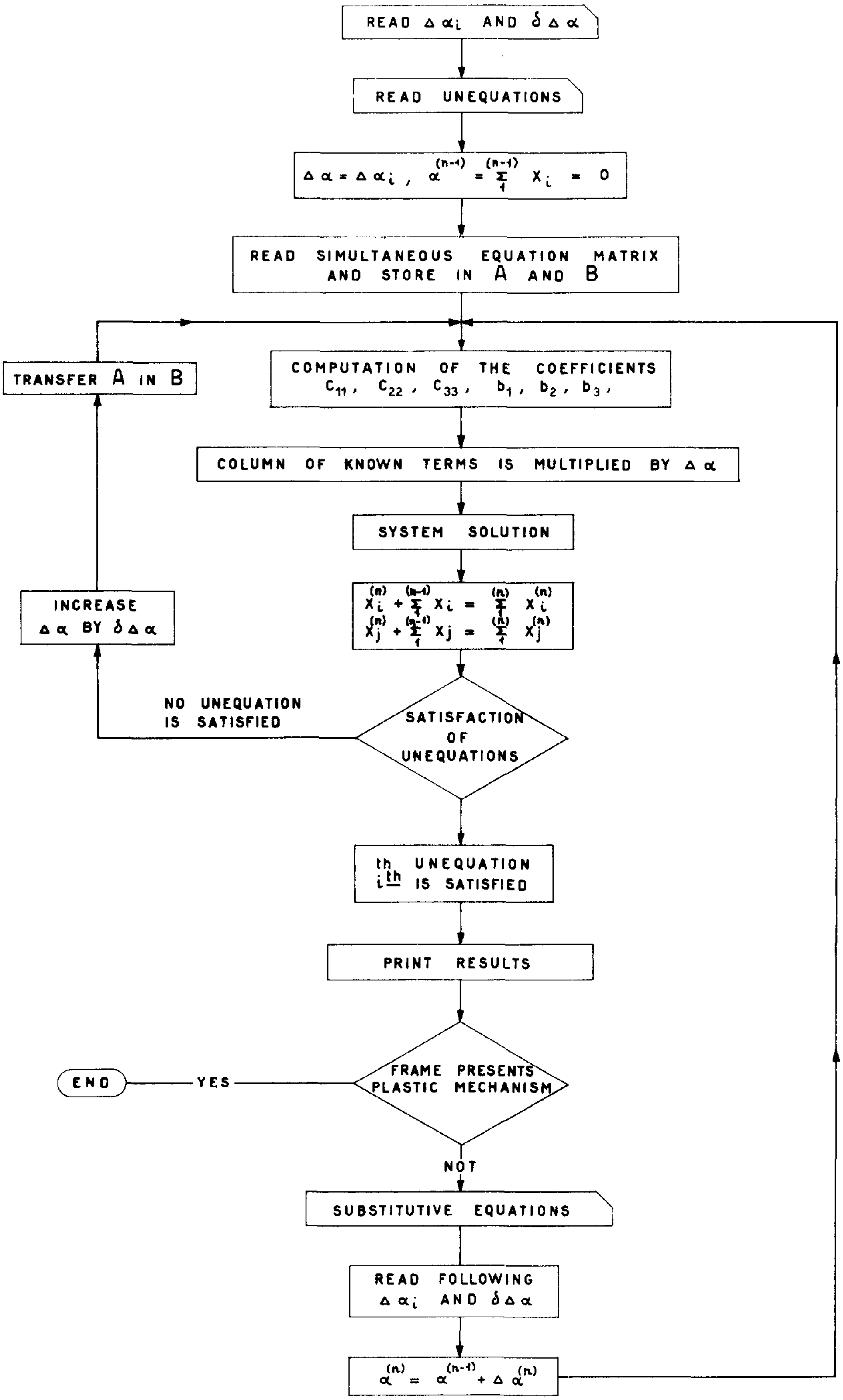

FIG. 7. Flow chart described in Section 7(b).

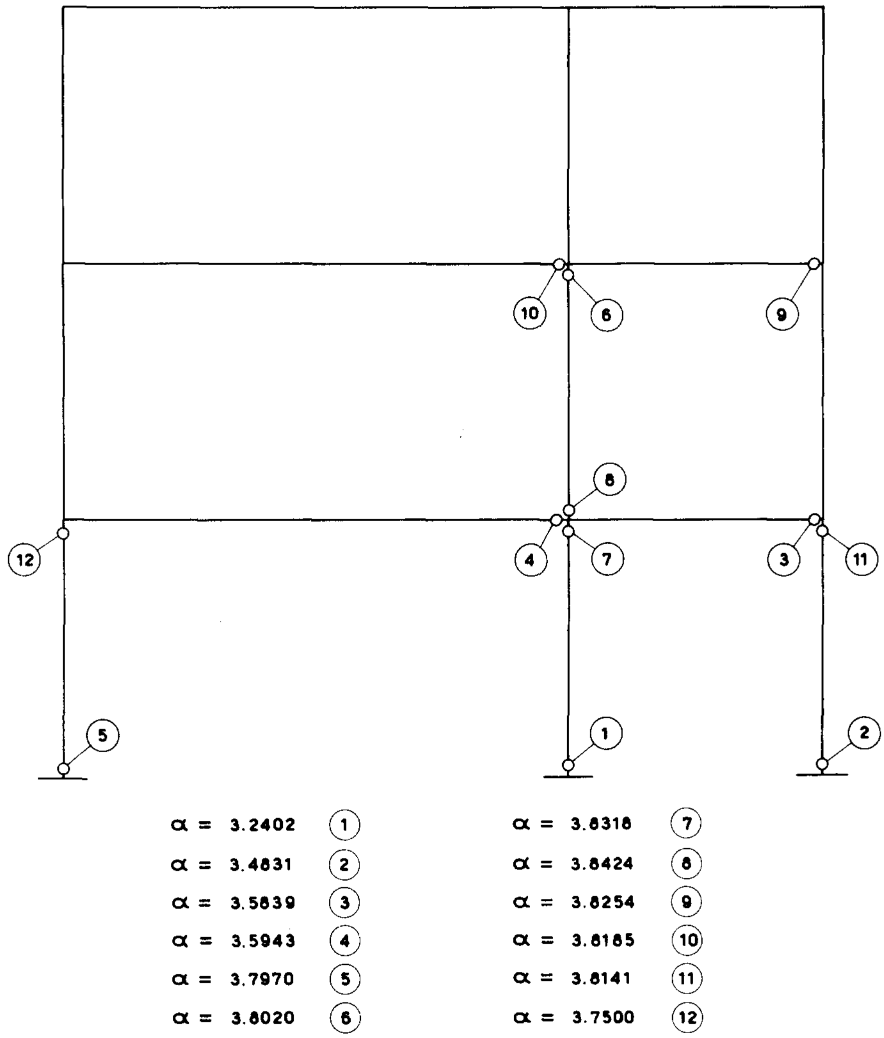

FIG. 8. Plastic hinges which are formed, according to the new procedure, in the configuration of the frame, which forms the model to be submitted to experimental test.

we have:

$$0{\cdot}027759\ \Delta\alpha^{(8)}(2610{\cdot}4428+X_1)+3{\cdot}8318\times 0{\cdot}027759X_1$$
$$+X_8+X_9+X_{10}+X_{11}+X_{12}+X_{13}+\Delta\alpha^{(8)}\times 400\times 2{\cdot}50 = 0$$

or:

$$0{\cdot}027759(\Delta\alpha^{(8)}+3{\cdot}8318)X_1+X_8+X_9+X_{10}+X_{11}+X_{12}+X_{13}$$
$$= -\Delta\alpha^{(8)}(1000+0{\cdot}027759\times 2610{\cdot}4428).$$

TABLE 1. TOTAL VALUES OF THE UNKNOWNS FOR EACH PHASE AS CALCULATED BY THE ELECTRONIC COMPUTER

UNKNOWNS	$\alpha = 3.2402$	$\alpha = 3.4831$	$\alpha = 3.5839$	$\alpha = 3.5943$	$\alpha = 3.7970$	$\alpha = 3.8020$
X_1	2151.1262	2304.9385	2370.5564	2378.1481	2553.0982	2559.9687
X_2	3578.0662	3846.4227	3972.5386	3991.0592	4527.1038	4557.2355
X_3	3153.8960	3477.4515	3699.3666	3728.4790	4520.3048	4591.6040
X_4	− 558.9690	− 602.0967	− 619.0336	− 620.7732	− 650.4889	− 651.0212
X_5	1875.2051	2013.7418	2071.8267	2077.8353	2199.3605	2202.5907
X_6	− 143.2124	− 157.9956	− 164.1473	− 164.5614	− 163.3036	− 164.6231
X_7	961.2252	1030.8873	1059.3063	1062.6326	1132.8376	1135.4385
X_8	558.9690	602.0967	619.0336	620.7732	650.4889	651.0212
X_9	604.0171	645.1255	659.3988	661.6775	716.7628	719.9374
X_{10}	− 1731.9926	− 1855.7462	− 1907.6793	− 1913.2738	− 2034.0569	− 2037.9676
X_{11}	− 1280.2839	− 1365.8377	− 1401.1636	− 1404.5064	− 1475.4456	− 1477.0317
X_{12}	961.2252	− 1030.8873	− 1059.3063	− 1062.6326	− 1132.8376	− 1135.4385
X_{13}	− 621.9121	− 670.4438	− 686.8539	− 689.0814	− 719.6809	− 720.6584
X_{14}	− 288.0666	− 314.1136	− 325.4105	− 325.4853	− 304.9092	− 301.6370
X_{15}	2349.4379	2525.1647	2598.5251	2607.0548	2791.1088	2797.9847
X_{16}	1099.5181	1182.2648	1221.3364	1226.5246	1365.0756	1372.0287
X_{17}	2091.5601	2241.6827	2310.7269	2318.5025	2517.7623	2525.9345
X_{18}	− 315.9504	− 331.0119	− 333.9883	− 336.1922	− 411.8535	− 418.3003
X_{19}	145.9738	191.3058	234.2414	237.9447	367.2615	371.8123
X_{20}	− 2168.6721	− 2341.5919	− 2418.6979	− 2429.0730	− 2680.7388	− 2692.9817
X_{21}	− 2081.8291	− 2265.1704	− 2340.7657	− 2349.4960	− 2470.6378	2472.6008
X_{22}	− 1469.6479	− 1571.2389	− 1623.8729	− 1629.4211	− 1798.0814	− 1805.2760
X_{23}	− 1229.7567	− 1290.9934	− 1331.3622	− 1330.4010	− 1315.6329	− 1307.4068
X_{24}	217.2791	246.1148	281.7853	288.4402	370.7228	378.8053
X_{25}	2667.2431	2862.1451	2960.5722	2973.7004	2973.7004	2973.7004
X_{26}	1513.0953	1614.1532	1669.7117	1673.4792	1908.1864	1926.4731
X_{27}	2658.1855	2877.9952	2973.6807	2973.6807	2973.6807	2973.6807
X_{28}	− 363.2530	− 437.4207	− 516.0267	− 526.3850	− 737.9844	− 750.6177
X_{29}	− 1758.5745	− 1957.4361	− 2107.6966	− 2127.4320	− 2629.1446	− 2628.7920
X_{30}	− 2098.5092	− 2210.5278	− 2289.5182	− 2297.6836	− 2411.2489	− 2427.5726
X_{31}	− 2626.2026	− 2600.0778	− 2589.3750	− 2588.0284	− 2568.7602	− 2568.5568
X_{32}	− 1428.4288	− 1587.0017	− 1642.3184	− 1643.2796	− 1658.0477	− 1666.2739
X_{33}	− 2291.1624	− 2532.2266	− 2518.9575	− 2517.7584	− 2489.7100	− 2488.6417

TABLE 1 (continued)

UNKNOWNS	$\alpha = 3.8318$	$\alpha = 3.8424$	$\alpha = 3.8254$	$\alpha = 3.8185$	$\alpha = 3.8141$	$\alpha = 3.7500$
X_1	2610.4428	2702.1206	2872.7718	2927.1426	2953.0834	3061.1193
X_2	4870.5229	5702.0279	7231.6109	7537.4000	7666.1779	8332.5844
X_3	5174.9934	7290.4146	10741.3140	11365.6110	11604.6250	14006.2210
X_4	− 656.1776	− 656.4625	− 647.7081	− 638.6918	− 634.0366	593.5760
X_5	2226.4011	2248.4620	2272.5904	2280.3005	2283.3521	2277.9470
X_6	− 149.9209	− 110.5575	− 27.4503	− 7.0039	2.8105	47.8356
X_7	1150.0290	1172.2198	1207.6640	1214.7384	1218.0737	1215.4810
X_8	656.1776	656.4625	647.7081	638.6918	634.0366	593.5760
X_9	769.4050	878.6116	1090.3838	1124.4999	1140.3604	1165.9695
X_{10}	− 2076.4802	− 2137.9044	− 2245.1400	− 2273.2966	− 2286.1626	− 2325.7826
X_{11}	− 1529.5716	− 1621.2532	− 1789.4685	− 1822.6416	− 1837.0767	− 1876.8861
X_{12}	− 1150.0290	− 1172.2198	− 1207.6640	− 1214.7384	− 1218.0737	− 1215.4810
X_{13}	− 702.6694	− 656.4163	− 550.8096	− 506.8777	− 486.1137	− 346.7193
X_{14}	− 269.4959	− 175.9877	22.8032	72.4437	90.8463	183.5617
X_{15}	2829.9235	2876.1877	2949.5998	2973.8595	2973.6323	2973.6323
X_{16}	1391.2559	1438.5483	1540.7470	1550.1313	1554.2247	1600.2621
X_{17}	2591.6570	2720.4781	2977.0442	2977.0442	2973.6445	2973.6445
X_{18}	− 499.9090	− 702.6239	− 1113.1870	− 1196.9436	− 1231.2067	− 1349.5312
X_{19}	377.6294	493.8987	599.4056	606.0620	603.9966	834.0396
X_{20}	− 2691.6078	− 2693.4828	− 2700.8783	− 2701.3492	− 2690.7803	− 2697.0083
X_{21}	− 2477.6167	− 2683.7848	− 2691.1802	− 2691.6511	− 2691.5818	− 2697.8097
X_{22}	− 1888.9876	− 2064.0617	− 2426.2345	− 2470.1664	− 2487.5308	− 2626.9252
X_{23}	− 1262.5855	− 987.6034	− 599.6745	− 531.0584	− 507.8135	− 504.4277
X_{24}	460.5464	770.1605	1319.2980	1424.6184	1467.2829	1821.2739
X_{25}	2973.7004	2973.7004	2973.7004	2973.7004	2973.7004	2973.7004
X_{26}	2071.8264	2280.3609	2291.1584	2291.4666	2291.4031	2303.7802
X_{27}	2973.6807	2973.6807	2973.6807	2973.6807	2973.6807	2973.6807
X_{28}	− 838.1759	− 1264.0592	− 1918.7037	− 2030.6804	− 2071.2796	− 2655.3136
X_{29}	− 2627.0493	− 2630.1531	− 2639.7023	− 2642.2177	− 2643.3127	− 2654.6011
X_{30}	− 2567.9101	− 2570.2765	− 2573.6785	− 2573.5159	− 2573.5216	− 2579.6708
X_{31}	− 2567.7240	− 2570.0904	− 2573.4924	− 2573.3298	− 2573.5555	− 2579.7047
X_{32}	− 1711.0952	− 1986.0772	− 2374.0061	− 2442.6222	− 2465.8672	− 2469.2530
X_{33}	− 2981.5365	− 2472.6227	− 2465.1941	− 2465.0829	− 2465.1916	− 2468.5773

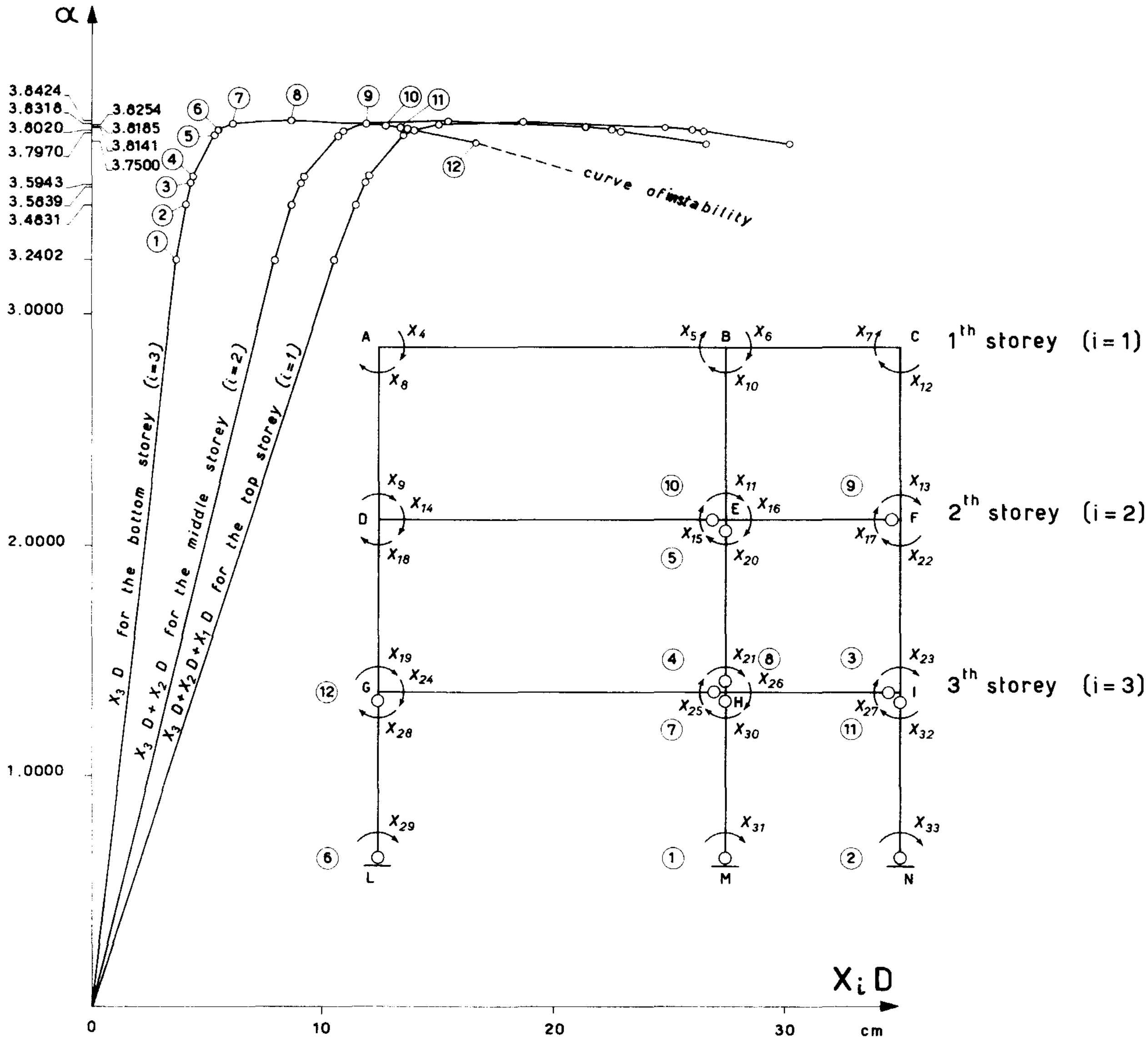

FIG. 9(a). Horizontal displacement Δ_i of three floors, as a function of α. The unknown bending moments, appearing in the equation system have been written on the frame configuration, the unknowns X_1, X_2, X_3 are the relative horizontal displacement of the storeys (regardless of factor D_i). On the other hand, the curves represent the absolute movements of the horizontal floors.

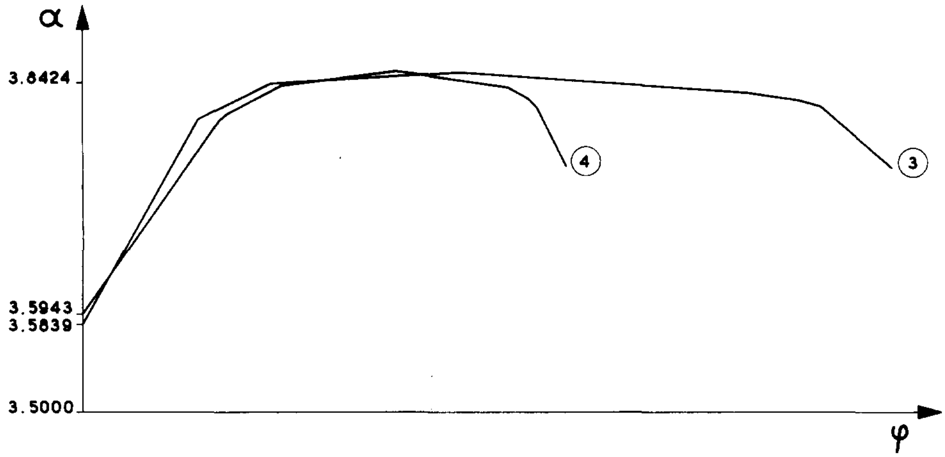

FIG. 9(b). For example, the rotation of the hinges in sections 25 and 27 are shown. The diagram is analogous for all the other hinges.

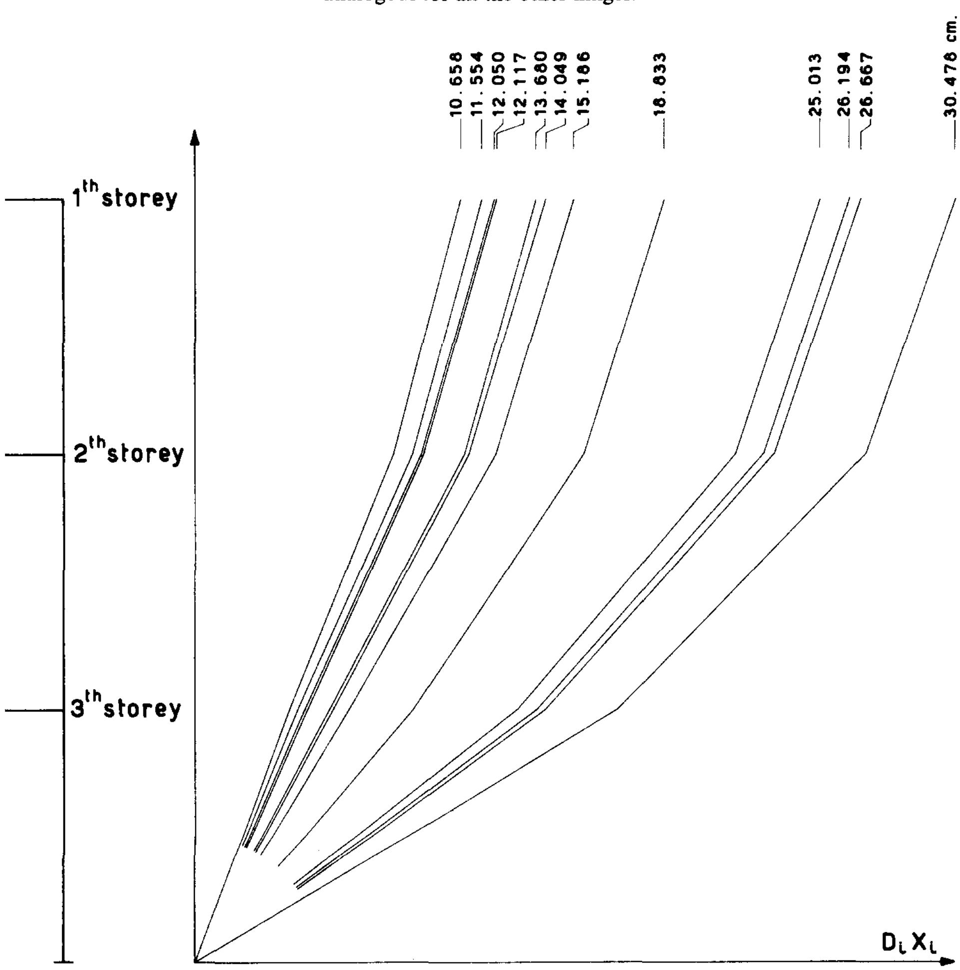

FIG. 10. Deformation of the frame in the twelve phases of the procedure.

Similarly, for the middle storey ($i = 2$) we have:

$$N_2 D = 0{\cdot}055158; \qquad \sum_1^7 X_2 = 4870{\cdot}5229 \text{ kg-m}$$

$$F_2 = 800 \text{ kg}; \qquad h_2 = 2{\cdot}50 \text{ m}.$$

Thus obtaining:

$$0{\cdot}055158(\Delta\alpha^{(8)} + 3{\cdot}8318)X_2 + X_{18} + X_{19} + X_{20} + X_{21} + X_{22} + X_{23}$$
$$= -\Delta\alpha^{(8)}(2000 + 0{\cdot}055158 \times 4870{\cdot}5229).$$

Finally, for the bottom storey ($i = 3$), we have:

$$N_3 D = 0{\cdot}082739; \qquad \sum_1^7 X_3 = 5174{\cdot}9934 \text{ kg-m};$$

$$F_3 = 1200 \text{ kg}; \qquad h_3 = 2{\cdot}50 \text{ m}.$$

and obtain:

$$0{\cdot}082739(\Delta\alpha^{(8)} + 3{\cdot}8318)X_3 + X_{28} + X_{29} + X_{30} + X_{31} + X_{32} + X_{33}$$
$$= -\Delta\alpha^{(8)}(3000 + 0{\cdot}082739 \times 5174{\cdot}9934).$$

The equations relative to the node G at which bars without plastic hinges at the ends join, are written symbolically:

$$\begin{cases} \sum_K X_j^{(8)} = 0 \\ \varphi_{\mathrm{GD}} = \varphi_{\mathrm{GH}} \\ \varphi_{\mathrm{GH}} = \varphi_{\mathrm{GL}} \end{cases}$$

In explicit form we obtain, respectively:†

$$\begin{cases} X_{19} + X_{24} + X_{20} = 0 \\ X_{18} - 2X_{19} + 4X_{24} - 2X_{25} - X_2 = -2670\,\Delta\alpha^{(8)} \\ X_{29} - 2X_{28} + 4X_{24} - 2X_{25} - X_3 = -2670\,\Delta\alpha^{(8)} \end{cases}$$

The equations referring to the node H, which, in this phase, develops hinges in the sections where X_{25} and X_{30} are applied, are also of type (1d). In symbolic form we have:

$$\begin{cases} \sum_K X_j^{(8)} = 0 \\ \varphi_{\mathrm{HE}} = \varphi_{\mathrm{HI}} \\ \Delta\sigma_{25} = 0 \\ \Delta\sigma_{30} = 0 \end{cases}$$

† The coefficient of $\Delta\alpha^{(8)}$, in the second term written, for the longer bars, has the value of $3 \times 2 \times \mu' = 3 \times 2 \times \frac{2}{9}$ $Pl = 2{\cdot}670$ kg-m. The factor 3 derives from the algebraic sum of the moments of fixing of the two ends of the bar due to external loads. Factor 2 depends on the fact that the length of the beam is twice the one taken as the unit of length. For the other bars the coefficient is $3 \times \mu'' = 3 \times \frac{1}{8}$ $Pl = 375$ kg-m. Factor 2 does not appear in this instance, since the length of the bar is precisely that of the control. For the stanchions the coefficient is nil.

We deduce:

$$\left\{\begin{aligned} &X_{21}+X_{25}+X_{26}+X_{30}=0\\ &X_{20}-2X_{21}+2X_{26}-2X_{27}-X_2=-375\,\Delta\alpha^{(8)}\\ &\frac{X_{25}}{0{\cdot}00009592}=0\\ &\frac{-X_{30}}{0{\cdot}00009592}+\frac{1}{0{\cdot}002126\times 5}(X_4+X_5+X_{14}+X_{15}+X_{24}+X_{25}-2X_6\\ &\qquad -2X_7-2X_{16}-2X_{17}-2X_{26}-2X_{27})+\Delta\alpha^{(8)}\cdot\frac{3{\cdot}000}{0{\cdot}002126}=0. \end{aligned}\right.$$

In the last equation the normal stress is expressed in terms of the moments X_j and of the external loads. The value $\delta\Delta\alpha = 0{\cdot}0001$ has been assigned to the phase under examination.

The eighth hinge forms in section 21 in which for $\Delta\alpha^{(8)} = 0{\cdot}0106$ the plasticization condition is satisfied and has the following expression:

$$-\frac{X_{21}}{0{\cdot}00009592}+\frac{1}{0{\cdot}002126\times 5}(X_4+X_5+X_{14}+X_{15}-2X_6-2X_7-2X_{16}-2X_{17})$$
$$+\Delta\alpha^{(8)}\frac{2{\cdot}000}{0{\cdot}002126}-31{,}000{,}000=0.$$

In the light of what is set out above, it is thought reasonable to state that the proposed theory appears to be applicable to rigid steel I-beam frames and that it could become a valid instrument for practical technical applications, with the help of an electronic computer.

It is felt that the hypotheses formulated are such as to supply theoretical results giving a greater margin of safety when compared with the results in practice. In this connection, the results of experiments still under way, will be communicated as soon as available.

Acknowledgement—We gratefully acknowledge the aid of Mr. F. Maceri, Engineer of the Electronic Computing Centre, in evolving the programmes.

A SELECTED BIBLIOGRAPHY

[1] M. Pagano, La rovina in regime elastoplastico ideale dei telai rigida in acciaio. *Costruz. metall.* n. 4 (1963).
[2] M. Pagano, *Strutture*, vol. II, parte II, edited by Liguori, Napoli (1963).
[3] J. F. Baker, The design of steel frames. *Struct. Engr* **27**, 397 (1949).
[4] M. R. Horne, Maximum beam moments in welded building frames. *Struct. Engr* **28**, 109 (1950).
[5] P. S. Symonds and B. G. Neal, Recent progress on the plastic methods of structural analysis. *J. Franklin Inst.* **252**, 383 (1951).
[6] P. S. Symonds and B. G. Neal, The interpretation of failure loads in the plastic theory of continuous beams and frames. *J. aeronaut. Sci.* **19**, 15 (1952).
[7] B. G. Neal and P. S. Symonds, The rapid calculation of the plastic collapse load for a framed structure. *Proc. Instn. civ. Engrs*, part III, **1**, 58 (1952).

[8] M. R. HORNE, The plastic model of British standard rolled steel joists. *Br. Weld. Res. Ass. Rep.* F.E. 1/33 (1953).
[9] W. MERCHANT, The failure load of rigid jointed frameworks as influenced by stability. *Struct. Engr* (July 1954).
[10] F. CAMPUS and C. MASSONET, Recherches sur le flambement de colonnes, en acier A 37, à profil en double té sollicitées obliquement. *Bull. Cent. Étud. Rech. scient., Liège.* **7**, 119 (1955) ou *C.r. Rech. Inst. Encour. scient. Ind. Agric.* n. 7.
[11] W. MERCHANT, Critical loads of tall building frames, part I. *Struct. Engr* **33**, 84 (1955).
[12] R. K. LIVESLEY, The application of an electronic digital computer to some problems of structural analysis. *Struct. Engr* **34**, 1 (1956).
[13] M. R. HORNE, The stanchion problem in frame structures designed according to ultimate carrying capacity. *Proc. Instn. civ. Engrs*, part III, **5**, 105 (1956).
[14] W. MERCHANT, Frame instability in the plastic range. *Symp.* on the *Plastic Theory of Structures*, Cambridge, 1956. *Br. Weld. J.* **3**, 366 (1946); also discussion by R. H. WOOD, p. 26.
[15] W. MERCHANT and R. B. L. SMITH, Critical loads of tall building frames, part II. *Struct. Engr* **34**, 284 (1956).
[16] R. E. BOWLES and W. MERCHANT, Critical loads of tall building frames, part III; IV. *Struct. Engr* **34**, 324 (1956); **36**, 187 (1958).
[17] M. R. HORNE, Multi-storey frames. *Br. Weld. J.* **3**, 336 (1956).
[18] J. F. BAKER, M. R. HORNE and J. HEYMAN, *The Steel Skeleton*, vol. II, plastic behaviour and design. Cambridge University Press (1956).
[19] R. K. LIVESLEY and D. B. CHANDLER, *Stability Functions for Structural Frameworks.* Manchester University Press (1956).
[20] H. G. ALLEN, The estimation of the critical loads of certain frameworks. *Struct. Engr* **35**, 135 (1957).
[21] R. KLEMENT, Knickuntersuchung von Rahmentragwerken nach DIN 4114 Ri 10.2 mit Hilfe des Angleichverfahrens von Kani. *Stahlbau* (Dec. 1957).
[22] M. PAGANO, Sul calcolo dei telai a nodi spostabili. *G. Genio civ.* fasc. 10 (1957).
[23] R. H. WOOD, The stability of tall buildings. *Proc. Instn. civ. Engrs* **7** (Sept. 1958).
[24] W. MERCHANT, C. A. RASHID, A. BOLTON and A. SALEM, *The Behaviour of Unclad Frames.* 15th Anniv. Conf. Instn. Struct. Engrs, London, 1958.
[25] A. SALEM, Frame instability in the plastic range. Ph.D. Thesis, Manchester University (May, 1958).
[26] M. W. LOW, Some model tests of multi-storey rigid steel frames. *Proc. Inst. civ. Engrs* **13** (1958).
[27] S. T. ARIARATNAM, The collapse load of elastic–plastic structures. Thesis presented to the University of Cambridge at Cambridge, England in 1959, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
[28] B. G. NEAL, *The Plastic Methods of Structural Analysis.* Chapman & Hall (1959).
[29] R. K. LIVESLEY, A programme for the elastic analysis of rigid-jointed plane structures for use on Edsac II. DC7/70 specifications. Cement and Concrete Assoc., London, England (1960).
[30] J. HEYMAN, An approach to the design of tall steel buildings. *Proc. Instn. civ. Engrs* **17**, 431 (1960).
[31] L. K. STEVENS, Direct design by limiting deformations. *Proc. Instn. civ. Engrs* **16**, 235 (1960).
[32] M. HOLMES, Steel frames with brickwork and concrete infilling. Civ. Engr. Inst. Tech. Bradford (July, 1960).
[33] M. PAGANO, Sul carico critico dei telai elastici multipiani. *Costruz. metall.* n. 2 (1960).
[34] W. J. AUSTIN, Strength and design of metal beam-columns. *J. Struct. Div. Am. Soc. civ. Engrs* **87**, No. ST4, Proc. Paper 2802, pp. 1–32 (April, 1961).
[35] R. K. LIVESLEY, Commentary on plastic design in steel. ASCE Manuals of Engineering Practice, Joint Committee of the Welding Research Council and ASCE No. 41 (1961).
[36] S. TIMOSHENKO and J. M. GERE, *Theory of Elastic Stability*, 2nd edition. McGraw-Hill (1961).
[37] J. HEYMAN, On the estimation of deflections in elastic-plastic frame structures. *Proc. Instn. civ. Engrs* **19**, 39 (1961).
[38] M. PAGANO and G. ZINGONE, Il calcolo di verifica dei telai piani alle soglie dell' instabilità. *Costruz. metall.* n. 3 (1961).
[39] Ch. MASSONET and M. SAVE, Calcul plastique des constructions. A.S.B.L. Bruxelles (1961).
[40] M. OJALVO and V. LEVI, Columns in planar continuous structures. *J. struct. Div. Am. Soc. civ. Engrs* **89**, No. ST1, Proc. Paper 3408, pp. 1–13 (Feb. 1963).
[41] N. C. LIND, Iterative limit load analysis for tall frames. *J. struct. Div. Am. civ. Engrs* **30**, part I (April 1964).
[42] M. R. HORNE, A moment distribution method for the analysis and design of structures by the plastic theory. Structural and Building Division Meeting, Dec. 1953.

(*Received* 11 *February* 1965; *revised* 20 *December* 1965)

Résumé—Cette étude traîte des résultats initiaux des recherches théoriques et expérimentales en ce qui concerne le comportement d'une structure de charpente en acier à rayon I sous un plan générique de poids horizontaux et verticaux proportionnellement variables.

Zusammenfassung—Diese Abhandlung beschäftigt sich mit Anfangsergebnissen von theoretischen und experimentalen Untersuchungen über das Verhalten von Stahlträgern I-Gerüsten auf Grund eines Schema von verhältnissgleichen veränderlichen senkrechten und waagrechten Belastungen.

Абстракт—Эта статья рассматривает основные результаты теоретического и экспериментального исследования, относящегося к поведению структуры конструкции стальной Iбалки под общей схемой пропорционально переменных вертикальных и горизонтальных нагрузок.